CR 134558

SHAPING FILTER REPRESENTATION OF NONSTATIONARY COLORED NOISE

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July 1968

Technical Report No. 106

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This work was supported in part by the National Aeronautics and Space Administration under Grant NGR-33-008-090 and by the National Science Foundation under Grant GK-2283.

Abstract

The problem of determining a shaping filter for nonstationary colored noise is considered. The shaping filter transforms white noise into a possibly nonstationary random process (having no white noise component) with a specified autocorrelation function. A set of conditions to be satisfied by an autocorrelation function leads to the determination of a shaping filter. The shaping filter coefficients are simply related to the solution of a matrix Riccati equation. In order to formulate the Riccati equation, basic results concerning the mean-square differentiability of a random process are developed. If the Riccati equation is undefined, an autonomous (zero-input) shaping filter may easily be determined.

Introduction

It is often convenient to model a random process as the result of a linear filtering operation on stationary white noise. Such a representation has proved invaluable when applied to many signal processing problems, especially those associated with the theory of filtering and estimation of random signals [1-6]. In such applications the given random signal or process is frequently specified only by its autocorrelation function. For example, optimal estimation problems involving a minimum mean-square error criterion invariably may be stated in terms of appropriate autocorrelation and cross-correlations of given random variables or processes [7]. In such cases, the statistical description of a random process given solely in terms of its second-order properties, i.e., its autocorrelation function, clearly suffices for the purpose of solving the estimation problem.

A more physical description of a random process with a given autocorrelation function employs a linear system called a shaping filter, which transforms stationary white noise into a process having the given autocorrelation function. By introducing shaping filters, great simplicity has been achieved in the formulation and solution of many estimation, filtering, and prediction problems. Wiener [1] used the shaping filter concept implicitly, and more recently Darlington [9], Kalman and Bucy [6], and others used it explicitly. Clearly, in order to understand the generality or the limitations of a shaping filter description of a random process, one must investigate the possibility of transforming a statistical description of the process into a shaping filter description. The so-called factorization problem, concerned with determining a shaping filter from a given autocorrelation function, is the primary subject of the present investigation.

For stationary random processes, the factorization problem has well-known frequency domain solutions. General solutions are given by Wiener [1], and Doob[8], and solutions for the case of a rational power spectral density are given by Wiener [1], and Bode and Shannon [2]. In the latter case, the shaping filters may be realized by the interconnection of a finite number of lumped elements. For purposes of simulation or computation, the latter case has considerable practical significance.

If the given autocorrelation function corresponds to a nonstationary random process, the factorization problem becomes particularly interesting and challenging. Although previous investigations of the nonstationary factorization problem have met with varying degrees of success, the problem has not been solved in general. A common assumption among these investigations, as well as the present one, is that the random processes under consideration are nonstationary analogues of those considered by Bode and Shannon. In other words, a shaping filter is represented either by a single linear differential equation of order n, or more generally, by a set of first-order linear differential equations in "state-variable" form with time-varying coefficients. In the treatment below, it will be assumed additionally that the shaping filter does not have a direct connection from input to output. This assumption means that the output random process is well behaved in the sense that it contains no white noise component. Thus the results are applicable specifically to state estimation problems with colored measurement noise.

Among the first and most significant contributions to the solution of the factorization problem is that of Darlington [9]. Darlington assumed the existence of a single n-th order differential equation model for the shaping filter. Using the algebra of time-varying differential operators and a method analogous to that employed by Bode and Shannon, he exhibited global solutions of the factorization problem, provided that the time variations involved were suitably defined and restricted. The coefficients of the shaping filter may be obtained from the solutions of a related linear differential equation.

Batkov [10], at about the same time as Darlington, proposed an algebraic solution of the factorization problem. However, according to Stear [11], Batkov's method fails except in certain special cases. R. P. Webb, et al. [12], considered a state-variable solution of the problem. Their solution too appears to be invalid except in special cases.

Other relevant investigations are those of Kalman [13], Stear [11], and Anderson [14], and are concerned with state-variable formulations. Although Kalman did not solve the factorization problem, he was able to establish a formal definition of the problem. The results of Stear and Anderson, although derived by different methods, are similar and appear to provide a significant first step in demonstrating the existence of a factorization for the general nonstationary case.

The present investigation may be regarded as an attempt to develop a realizability theory for shaping filters having a state-variable representation. Conditions are developed which, if satisfied by an autocorrelation function, guarantee that a shaping filter may be determined.

Problem Formulation

The class of shaping filters to be considered includes those which may be represented by the set of linear differential equations*

$$\dot{x}(t) = \beta(t)\, u(t)$$
$$y(t) = \phi^t(t)\, x(t) \tag{1}$$

The input u(t) is a scalar representing a zero mean white noise process, so that

$$E[u(t)\, u(\tau)] = \delta(t - \tau).$$

The state x(t) is an n-vector and the output y(t) a scalar. The coefficients $\phi(t)$ and $\beta(t)$ are real-valued n-vectors. We assume that the shaping filter corresponding to (1) is causal, i.e., non-anticipative. The representation (1) is completely general in the sense that any set of differential equations in "state-variable" form may be transformed to (1) by a suitable change of coordinates. Although the absence of a feedback matrix in (1) may make this form unsuitable for practical simulation, the theory of equivalent systems [15] is sufficiently developed to indicate when the above system has an equivalent but practical realization.

In what follows, it should be assumed that the initial value of the state variable $x_o = x(t_o)$ is derived from a zero-mean random

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* The superscript t will be used to denote matrix transposition and the symbol $\dot{z}(t)$ the first derivative of z(t). When the context is clear, the explicit dependence of a function on its argument will be suppressed.

variable uncorrelated with the white noise input u(t). The output y(t) is then a real-valued, zero-mean random process with autocorrelation function

$$r(t,\tau) = E[y(t)y(\tau)]. \tag{2}$$

For purposes of analysis, one may calculate $r(t,\tau)$ in terms of the coefficients $\phi(t)$ and $\beta(t)$ in a straightforward way. In terms of a matrix M(t) defined by

$$M(t) = E[x_o x_o^t] + \int_{t_o}^{t} \beta(\lambda)\beta^t(\lambda)d\lambda \,, \tag{3}$$

the autocorrelation function is

$$r(t,\tau) = \phi^t(t)M(\tau)\phi(\tau) \,, \tag{4a}$$

for $t > \tau$. For $\tau > t$, the above development may be repeated to yield

$$r(t,\tau) = \phi^t(t)M(t)\phi(\tau) . \tag{4b}$$

Thus, combining (4a) and (4b), we have

$$r(t,\tau) = \phi^t(t)M[\min(t,\tau)]\, \phi(\tau) \tag{5}$$

One may easily verify that the matrix M(t) is the covariance matrix of the state vector, i.e.,

$$M[\min(t,\tau)] = E[x(t)x^t(\tau)], \tag{6}$$

where x(t) is the state vector of the shaping filter. The salient properties of M(t) are stated in the following Lemma, which generalizes a result of Doob [16].

Lemma 1. Let a covariance matrix M be defined as in equation (6). Then M is symmetric and

(a) $M(t) \geq 0$, for all t .

(b) $M(t_2) - M(t_1) \geq 0$, for all $t_2 \geq t_1$.*

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* For real symmetric matrices A and B, the matrix inequality $A \geq B$ means that the matrix (A-B) is non-negative definite.

Proof: That $M(t) \geq 0$ and is symmetric follows by inspection of (6). Part (b) is established as follows:

$$0 \leq E[x(t_2) - x(t_1)][x(t_2) - x(t_1)]^t$$

$$= E[x(t_1)x^t(t_1)] + E[x(t_2)x^t(t_2)] - E[x(t_1)x^t(t_2)] - E[x(t_2)x^t(t_1)]$$

$$= M(t_2) - M(t_1).$$

The last equality follows from (6).

Note that $M(t)$ as defined by equation (3) is differentiable, so that from the Lemma, $\dot{M}(t) \geq 0$.

Definition 1. A symmetric, differentiable, real-valued matrix $M(t)$ will be called admissible if $M(t)$ is non-negative definite and non-decreasing.

The development above makes clear that admissible matrices will play a crucial role in what follows. According to (5), the function $r(t,\tau)$ bears a simple relation to the state variance matrix $M(t)$. This relation is used in the following Theorem, in which several important properties of $r(t,\tau)$ are derived.

Theorem 1.* Let the relation

$$r(t,\tau) = \phi^t(t)M[\min(t,\tau)]\phi(\tau) \ ,$$

* Theorems 1,3, and 5 were stated without proof in [22].

where $M(t)$ is an admissible matrix, define a function $r(t,\tau)$. Then $r(t,\tau)$ satisfies the following:

A1. $r(t,\tau)$ is separable; i.e., there exist real-valued vectors $\phi(t)$ and $\gamma(t)$ such that

$$r(t,\tau) = \begin{cases} \phi^t(t)\gamma(\tau), \text{ for } t \geq \tau \\ \phi^t(\tau)\gamma(t), \text{ for } t < \tau , \end{cases} \tag{7}$$

A2. $r(t,\tau)$ is symmetric; i.e.,

$$r(t,\tau) = r(\tau,t) ,$$

and

A3. $r(t,\tau)$ is non-negative definite; i.e., for any choice of instants $t_1 \leq t_2 \leq \ldots t_m$, for any choice of scalars $\alpha_1, \alpha_2, \ldots \alpha_m$, and for any finite integer m, the following quadratic form is nonnegative:

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i r(t_i,t_j)\, \alpha_j \geq 0 .$$

Proof: The first assertion follows by equating

$$\gamma = M\, \phi .$$

Symmetry is apparent by inspection of (5). In order to prove the third assertion, define matrices $\Delta_1, \Delta_2, \ldots \Delta_m$ as

$$\Delta_1 = M(t_1) \geq 0$$

$$\Delta_k = M(t_k) - M(t_{k-1}) \geq 0, \text{ for } k=2, \ldots m .$$

Then Q becomes

$$Q = \sum_{k=1}^{m} \sum_{i=k}^{m} \sum_{j=k}^{m} \alpha_i \phi^t(t_i) \Delta_k \phi(t_j) \alpha_j$$

$$= \sum_{k=1}^{m} \| \sum_{i=k}^{m} \alpha_i \Delta_k^{1/2} \phi(t_i) \|^2 \geq 0 ,$$

where $\|\cdot\|$ denotes the Euclidean norm, and $\Delta_k^{1/2}$ a real-valued square-root matrix of Δ_k.

It is well known [17] that an arbitrary function $r(t,\tau)$ is an autocorrelation function if and only if $r(t,\tau)$ satisfies A2 and A3 of the above Theorem. A1 reflects the fact that the shaping filter has a finite-dimensional state space. We assume henceforth that $r(t,\tau)$ satisfies A1-A3.

In order that a shaping filter corresponding to $r(t,\tau)$ exists, it is necessary and sufficient that (5) be satisfied for some admissible matrix $M(t)$ for which rank $\dot{M}(t) \leq 1$. Necessity has already been demonstrated in the calculation leading to (5). To prove sufficiency, note that the shaping filter coefficient $\beta(t)$ may be determined from M since $\beta\beta^t = \dot{M}$. A random initial value of the state at $t=t_o$ may be chosen from an ensemble for which $E[X(t_o)X^t(t_o)] = M(t_o)$.

Since $r(t,\tau)$ satisfies A1, we may equate

$$\gamma(t) = M(t)\,\phi(t) \tag{8}$$

to within a constant linear transformation taken as identity for convenience. Equation (8) may be regarded as the basic equation to which an admissible matrix solution M(t) must be sought in order to obtain the desired shaping filter.

The Derivative of a Random Process

Some new results concerning the existence of a derivative of a random process will be presented in this section. These results are pertinent because a parameter defined shortly associated with differentiability of the process is used in determining the coefficients of the shaping filter.

The concept of mean-square differentiation is defined below.

Definition 2. Let $y(t)$ be a random process for which $E[y^2(t)] < \infty$ for all t in T, an interval of interest. The process $y(t)$ has a derivative in the mean-square sense, denoted by $\dot{y}(t)$ or $y^{(1)}(t)$, at a point t in T if

$$\underset{h \to 0}{\text{l.i.m.}} \frac{y(t+h) - y(t)}{h} = \dot{y}(t),$$

where $t+h$ is in T, and l.i.m. denotes limit in the mean square sense. For brevity, the process $\dot{y}(t)$ will simply be called the derivative of $y(t)$.

A well-known condition for the existence of a derivative of a random process is stated in the following Theorem. The proof, due to Loève [17] is omitted.

<u>Theorem 2</u>. $y(t)$ has a derivative at t in T if and only if the function

$$\frac{\partial^2}{\partial t \partial \tau} r(t,\tau)$$

exists and is finite at the point (t,t).

Theorem 2 is generally difficult to apply. We derive below a more easily applicable differentiation criterion. Assume that the vectors ϕ and γ given in A1 are continuously differentiable, and define a function

$$d_o^2(t) = \phi^t(t)\,\dot{\gamma}(t) - \dot{\phi}^t(t)\,\gamma(t). \tag{9}$$

This expression may be rewritten as

$$d_o^2(t) = \lim_{t' \to t} \frac{\phi^t(t')[\gamma(t') - \gamma(t)] - [\phi(t') - \phi(t)]^t\,\gamma(t)}{t' - t}$$

$$= \lim_{t' \to t} \left[\frac{r(t',t') - r(t',t) - r(t,t') + r(t,t)}{t' - t}\right]$$

$$= \lim_{t' \to t} \frac{E[\{y(t') - y(t)\}^2]}{t' - t} \geq 0. \tag{10}$$

The following Theorem establishes the differentiation criterion.

Theorem 3. Let $y(t)$ be a random process having an autocorrelation function satisfying A1 - A3 and let $d_o^2(t)$ be defined by (9). Then $y(t)$ is differentiable at a point t if and only if $d_o^2(t) = 0$.

Proof. To prove the "only if" part, note that if $\dot{y}(t)$ exists, then

$$E[\dot{y}^2(t)] = \lim_{t' \to t} \frac{E[(y(t') - y(t))^2]}{(t' - t)^2} < \infty$$

from Theorem 2, and from (10) it is clear that $d_o^2(t) = 0$. To prove the "if" part consider the continuous function $\frac{\partial}{\partial t} r(t,\tau)$ with t fixed and τ varying. From A1,

$$\frac{\partial}{\partial t} r(t,\tau) = \begin{cases} \dot{\phi}^t(t)\ \gamma(\tau) & \text{for } t \geq \tau \\ \phi^t(\tau)\ \dot{\gamma}(t) & \text{for } t < \tau. \end{cases}$$

If $d_o^2(t) = 0$, then $\frac{\partial}{\partial t} r(t,\tau)$ is a continuous function of τ at the point $\tau = t$. Since the functions $\phi(t)$ amd $\gamma(t)$ are continuously differentiable, the function $\frac{\partial}{\partial t} r(t,\tau)$ has both a left-hand and right-hand derivative with respect to τ at the point $\tau = t$. These derivatives are equal and their common value is

$$\lim_{\tau \to t} \frac{\partial^2}{\partial t \partial \tau} r(t,\tau) = \dot{\phi}^t(t)\ \dot{\gamma}(t) < \infty .$$

Then from Theorem 2, $\dot{y}(t)$ exists.

Theorem 3 has an important corollary which will be used in the sequel. This corollary requires index functions $d_i^2(t)$ to be defined as follows:

$$d_i^2(t) = \phi^{(i)t}(t)\ \gamma^{(i+1)}(t) - \phi^{(i+1)t}(t)\ \gamma^{(i)}(t).^{*} \tag{11}$$

Corollary 1. Let the functions ϕ and γ possess at least k continuous derivatives. Then a random process $y(t)$ has a k-th derivative $y^{(k)}(t)$ at a point t if and only if the functions $d_0^2(t)$, $d_1^2(t)$, $\cdots$ $d_{k-1}^2(t)$ are all zero at t. In addition, if $k+1$ derivatives of ϕ and γ exist, thus allowing $d_k^2(t)$ to be defined, the inequality $d_k^2(t) \geq 0$ is valid at the point t. Finally,

$$E\left[y^{(i)}(t)\ y^{(j)}(\tau)\right] = \frac{\partial^{(i+j)}}{\partial t^i\ \partial \tau^j}\, r(t,\tau) \qquad \text{for all} \quad 0 \leq i,j \leq k.$$

Proof. The first and second assertions follow from the previous theorems and by induction on k. The last assertion follows from a Theorem of Loeve ([17], p. 471).

We shall assume for the remainder of the paper that $r(t,\tau)$ satisfies the hypotheses of the above Corollary as well as A1-A3 of Theorem 1; that is, $r(t,\tau)$ satisfies:

A4. The given functions $\phi(t)$ and $\gamma(t)$ possess at least $k+1$ continuous derivatives on T, the interval of interest.

A5. $d_i(t) = 0$ for all t in T and for $0 \leq i \leq k-1$, and $d_k(t) \neq 0$ for all t in T.

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* The k-th derivative of a random process or function $f(t)$ is denoted by $f^{(k)}(t)$.

Furthermore, T is to be considered an open interval so that desired derivatives will exist at interior points of T.

As a notational convenience we shall define the following matrices:

$$\left[R_k(t)\right]_{ij} = \frac{\partial^{(i+j)}}{\partial t^i \, \partial \tau^j} r(t,\tau)\bigg|_{\tau=t} ,$$

$$\Phi_k(t) = \left[\phi^{(0)}(t) \,\vdots\, \phi^{(1)}(t) \,\vdots\, \cdots \,\vdots\, \phi^{(k)}(t)\right] ,$$

$$\Gamma_k(t) = \left[\gamma^{(0)}(t) \,\vdots\, \gamma^{(1)}(t) \,\vdots\, \cdots \,\vdots\, \gamma^{(k)}(t)\right] ,$$

and

$$Y_k(t) = \mathrm{col}\left[y^{(0)}(t),\ y^{(1)}(t),\ \cdots\ y^{(k)}(t)\right] .$$

The previous Corollary implies that R_k is symmetric, non-negative definite, and that

$$R_k = E\left[Y_k \, Y_k^{\,t}\right] = \Phi_k^{\,t} \, \Gamma_k^{\,t} = \Gamma_k^{\,t} \, \Phi_k . \tag{12}$$

An important property, to be used shortly, is that $R_k(t)$ is non-singular. We establish this result by using the following Lemma.

<u>Lemma 2</u>. Let $r(t,\tau)$ satisfy A1-A3 with ϕ and γ m+1 times differentiable. If $d_m^{\,2}(t_1) > 0$ for some t_1 in T and $d_i^{\,2}(t) \equiv 0$ for $0 \le i \le m-1$ and for t in a neighborhood containing t_1, then $r(t_1,t_1) > 0$.

<u>Proof</u>. The Lemma is proved by contradiction. Let $r(t_1,t_1) = 0$. A3 implies $r(t,t_1) = r(t_1,t) = 0$ for all t in T. Therefore

$$\frac{d^i}{dt^i} r(t,t_1) = \frac{d^i}{dt^i} r(t_1,t) = 0 ,$$

or

$$\left[\phi^t(t)\ \gamma(t_1)\right]^{(i)} = \left[\phi^t(t_1)\ \gamma(t)\right]^{(i)} = 0 \tag{13}$$

for all t in T. For the case $m = 0$, (13) implies $d_o^2(t_1) = 0$. For the case $m = 1$, the equality

$$\dot{\phi}^t(t)\ \gamma(t) = \phi^t(t)\ \dot{\gamma}(t)$$

may be differentiated to yield

$$\phi^{(2)t}(t)\ \gamma(t) = \phi^t(t)\ \gamma^{(2)}(t) \tag{14}$$

for t in a neighborhood of t_1. Differentiating (14) and evaluating the result at $t = t_1$ yields

$$\left[\phi^t(t)\ \gamma(t_1)\right]^{(3)}\Big|_{t=t_1} + \phi^{(2)t}(t_1)\ \gamma^{(1)}(t_1)$$

$$= \phi^{(1)t}(t_1)\ \gamma^{(2)}(t_1) + \left[\phi^t(t_1)\ \gamma(t)\right]^{(3)}\Big|_{t=t_1}$$

which implies from (13) and (11) that $d_1^2(t_1) = 0$. The case for general m is proved in a straightforward way using the preceding differentiation argument.

We now show that R_k is nonsingular.

Theorem 4. Let $r(t,\tau)$ satisfy A1-A5 on the open interval T. Then the matrix R_k is positive definite everywhere in T.

Proof. Assume that R_k is singular at $t = t_1$. Then for some $\ell \le k$, and for some set of scalars $a_0, \cdots a_\ell$ with $a_\ell \neq 0$ the random process z(t) defined as

$$z(t) = \sum_{i=0}^{\ell} a_i \, y^{(i)}(t)$$

vanishes at $t = t_1$ with probability one. Define $\underline{r}(t,\tau)$ as $\underline{r}(t,\tau) = E[z(t)\, z(\tau)]$, which satisfies the hypotheses of Lemma 2 with $m = k - \ell$. But from Lemma 2

$$0 = \left. \frac{\partial^{2m+1}}{\partial t^m \, \partial\tau^{m+1}} \, r(t,\tau) - \frac{\partial^{2m+1}}{\partial t^{m+1} \, \partial\tau^m} \, r(t,\tau) \right|_{t=\tau=t_1} = a_\ell^{\,2} \, d_k^{\,2}(t_1) \; ,$$

which implies $a_\ell = 0$, a contradiction. Hence R_K is nonsingular and from (12) positive definite everywhere in T.

Derivation of a Shaping Filter

We shall use the results of the previous section to determine an admissible solution M(t) of the basic equation (8). The following Theorem shows that M(t) may be obtained as a solution of a matrix Riccati equation.

Theorem 5.* Let $r(t,\tau)$ satisfy conditions A1-A5. Then the following assertions are valid.

(i) If $\gamma(t) = M(t)\, \phi(t)$ on T, where M(t) is symmetric and the rank of $\dot{M}(t) \leq 1$, then M(t) satisfies the following Riccati differential equation:

$$\dot{M} = \frac{(\gamma^{(k+1)} - M\phi^{(k+1)})(\gamma^{(k+1)} - M\phi^{(k+1)})^t}{d_k^{\,2}} \; . \tag{15}$$

* Although derived independently, the methods used to obtain the results stated in Theorem 5 are essentially the same as those employed by Anderson [14], who obtained similar results for the special case $k = 0$.

(ii) Let M_o be any symmetric non-negative definite matrix which satisfies

$$\Gamma_k(t_o) = M_o \, \Phi_k(t_o) \tag{16}$$

for some t_o in the open interval T. If $M(t)$ is the solution of equation (15) having the initial value $M(t_o) = M_o$, then $M(t)$ is admissible and $\Gamma_k(t) = M(t)\,\Phi_k(t)$ for all $t > t_o$ in a neighborhood of t_o. Furthermore, the coefficient $\beta(t)$ may be evaluated as

$$\beta(t) = \frac{\gamma^{(k+1)}(t) - M(t)\,\phi^{(k+1)}(t)}{d_k(t)}. \tag{17}$$

Proof. Part (i) is proved by induction. By assumption $\gamma = M\phi$ and $\dot{M} = \beta\beta^t$ where β is an n-vector. Assume $\gamma^{(i)} = M\phi^{(i)}$. Differentiating this expression and pre-multiplying by $\phi^{(i)t}$ yields

$$\phi^{(i)t}\,\gamma^{(i+1)} = \phi^{(i)t}\,M\,\phi^{(i+1)} + \left[\phi^{(i)t}\,\beta\right]^2 .$$

But $\phi^{(i+1)t}\,\gamma^{(i)} = \phi^{(i+1)t}\,M\,\phi^{(i)} = \phi^{(i)t}\,M\,\phi^{(i+1)}$. Hence $\phi^{(i)t}\,\beta = d_i$ and $\gamma^{(i+1)} = M\,\phi^{(i+1)}$. Thus, by induction, $\gamma^{(i)} = M\,\phi^{(i)}$ for $0 \le i \le k$. For $i = k$ we have from the preceding argument

$$\gamma^{(k+1)} = M\,\phi^{(k+1)} + \beta\,d_k$$

or

$$\beta = (\gamma^{(k+1)} - M\,\phi^{(k+1)})/d_k .$$

Post-multiplying this expression by its transpose yields the desired Riccati equation.

The assertion in part (ii) will be proved by exhibiting a homogeneous linear differential equation, the solution of which is a vector of dimension n(k+1) which has as components the columns of the matrix $\Gamma_k - M \Phi_k$.

Consider the right member the following identity:

$$\frac{d}{dt}(\Gamma_k - M \Phi_k) = (\dot{\Gamma}_k - M \dot{\Phi}_k) - \dot{M} \Phi_k .$$

Substituting equation (15) for $\dot{M}$ yields

$$(\dot{\Gamma}_k - M \dot{\Phi}_k) - \frac{(\gamma^{(k+1)} - M \phi^{(k+1)})(\gamma^{(k+1)t} \Phi_k - \phi^{(k+1)t} M \Phi_k)}{d_k^2} .$$

If the quantity $\phi^{(k+1)t} \Gamma_k$ is added and subtracted in the right-most parentheses, the following expression results:

$$(\dot{\Gamma}_k - M\dot{\Phi}) - \frac{(\gamma^{(k+1)} - M\phi^{(k+1)})(\gamma^{(k+1)t} \Phi_k - \phi^{(k+1)t} \Gamma_k + \phi^{(k+1)t}(\Gamma_k - M \Phi_k))}{d_k^2} .$$

Let e_1 represent the (k+1)-dimensional unit vector with unity in the (k+1)-st position and zero elsewhere. Then the last expression becomes

$$(\dot{\Gamma}_k - M \dot{\Phi}) - (\gamma^{(k+1)} - M\phi^{(k+1)})\left[e_1^t + \frac{\phi^{(k+1)t} (\Gamma_k - M \Phi_k)}{d_k^2}\right],$$

which may be expanded as

$$\frac{d}{dt}(\Gamma_k - M \Phi_k) = \left[\dot{\Gamma}_k - M \dot{\Phi}_k - (\gamma^{(k+1)} - M \phi^{(k+1)}) e_1^t\right]$$

$$- \frac{(\gamma^{(k+1)} - M \phi^{(k+1)}) \phi^{(k+1)t}}{d_k^2} (\Gamma_k - M \Phi_k) . \quad (18)$$

Define a matrix A as

$$A = -\frac{(\gamma^{(k+1)} - M\phi^{(k+1)})\ \phi^{k+1)t}}{d_k^2},$$

and let q_i denote the i-th column of the matrix $\Gamma_k - M\Phi_k$. From the standard existence theorem for ordinary differential equations [18], the matrix A defined in terms of M exists in a neighborhood of the point $t = t_o$. In terms of A and q_i, (18) may be re-written as the following set of differential equations.

$$\begin{aligned} \dot{q}_o &= Aq_o + q_1 \\ \dot{q}_1 &= Aq_1 + q_2 \\ &\vdots \\ \dot{q}_i &= Aq_i + q_{i+1} \\ &\vdots \\ \dot{q}_k &= Aq_k \end{aligned} \tag{19}$$

According to the hypothesis of the Theorem, the initial condition for (19) is $q_i(t_o) = 0$. Since (19) is linear, it has the unique solution $q_i(t) = 0$ for all t in the neighborhood of t_o for which the matrix $M(t)$ exists and for all $0 \le i \le k$.

The matrix $M(t)$ is admissable because M_o is non-negative definite and symmetric, and $\dot{M}$ may be expressed as an outer product $\beta\beta^t$, where β, from (17), is real-valued.

Therefore, the matrix $M(t)$, obtained as a solution of the Riccati equation, satisfies the basic equation (8). The coefficient vector β is evaluated from (17). In order to complete the description of the shaping filter, the existence of an initial covariance matrix Mo satisfying (16) must be verified.

Let Δ_o be any covariance matrix satisfying $\Delta_o \Phi_k(t_o) = 0$. Such a Δ_o always exists. From Theorem 4 $R_k(t_o)$ is nonsingular. Let

$$M_o = \Gamma_k(t_o)R_k^{-1}(t_o)\Gamma_k^t(t_o) + \Delta_o \quad (20)$$

Matrix M_o so defined certainly satisfies (16). Moreover it is easy to show, as was done in [19], that any matrix M_o satisfying (16) may be expressed as in (20). Therefore $\Gamma_k(t_o)R_k^{-1}(t_o)\Gamma_k^t(t_o)$ is the smallest matrix (in terms of the associated quadratic form) satisfying (16).

Theorem 5 states that the covariance of the state variable of a shaping filter must satisfy the Riccati equation (15). Conversely, any solution of (15) with initial value satisfying (16) yields a shaping filter corresponding to the given autocorrelation function. Hence the equations (15) and (17) provide a prescription for determining the shaping filter coefficients.

However, solutions of (15) are guaranteed to exist only in a local neighborhood of the initial time t_o. It is possible that a solution of (15) may become unbounded at a finite time $t_1 > t_o$. Clearly, such a solution yields a shaping filter defined only in the finite interval $t_o \le t < t_1$. This phenomenon is examined in the following example.

Let

$$r(t,\tau) = \frac{4}{3} e^{-|t-\tau|} - \frac{5}{12} e^{-2|t-\tau|} , \tag{21}$$

and let

$$\gamma(t) = \begin{bmatrix} \frac{4}{3} e^{t} \\ \\ \frac{-5}{12} e^{2t} \end{bmatrix} \text{ and } \phi(t) = \begin{bmatrix} e^{-t} \\ \\ e^{-2t} \end{bmatrix}.$$

In this example, n=2 so that matrix M(t) has order 2. However, by regarding the basic equation (8) as a linear constraint on M(t), equation (15) may be transformed into a scalar equation of the Riccati type. The new equation may be solved for the scalar $M_s(t)$, the (2,2) element of the matrix M(t).* The scalar Ricatti

* In the general case the Riccati equation (15), of order n, may be reduced to one of order n-k-1, [19].

equation corresponding to (21) is

$$\dot{M}_s(t) = \left(\frac{5}{4} e^{2t} - M_s(t)\, e^{-2t}\right)^2 . \tag{22}$$

The general solution of (22), illustrated in Figure 1, is

$$M_s(t) = \frac{\frac{25}{4} e^{4t} - \frac{1}{4}\left(\frac{M_s(0) - 25/4}{M_s(0) - 1/4}\right) e^{10t}}{1 - \left(\frac{M_s(0) - 25/4}{M_s(0) - 1/4}\right) e^{6t}} \tag{23}$$

for $t_o = 0$. If $M_s(0) > 25/4$, the denominator of (23) will vanish at a finite time $t_1 > 0$ so that $M_s(t)$ will be unbounded as t approaches t_1. If $M_s(0) \leq 25/4$, then $M_s(t)$ is well behaved.

Two solutions of (22) are of special interest. The solution

$$M_{s1}(t) = 25\ e^{4t}/4$$

corresponds to $M_{s1}(0) = 25/4$. The solution

$$M_{s2}(t) = e^{4t}/4$$

corresponds to $M_{s2}(0) = 1/4$. Evidently from (23),

$$\lim_{t\to\infty} M_s(t) = e^{4t}/4$$

for all $M_s(0) < 25/4$. Therefore the solution $M_{s1}(t)$ is unstable in the sense of Liapunov and represents a separatrix, and the solution $M_{s2}(t)$ is asymptotically stable. Note that since (21) corresponds to a stationary random process, shaping filters may also be determined by the method of Bode and Shannon [2], which yields the following two transfer functions for time-invariant shaping filters:

$$H_1(s) = \frac{s-3}{(s+1)(s+2)} ,$$

and

$$H_2(s) = \frac{s+3}{(s+1)(s+2)} .$$

These transfer functions correspond to the special solutions $M_{s1}(t)$ and $M_{s2}(t)$ respectively. Thus the time-domain solution for $M_s(t)$ shown in Figure 1 reveal special properties of the classical frequency-domain solution for the transfer function $H(s)$.

The finite escape time phenomenon illustrated in Figure 1 appears to be characteristic of problems involving a Riccati equation. Some recent works of Bucy [20] and Moore and Anderson [21] report sufficient conditions for solutions of a Riccati equation to be well defined in the future.

A Singular Case

The definition of the Riccati equation in the previous section requires that for some integer k, $d_k(t) \neq 0$ for all t in T. However a shaping filter may be determined in cases for which no such integer exists. We show here that if $d_i^2(t) \equiv 0$ on T for $0 \leq i \leq n-1$, then the basic equation (8) is satisfied by a constant matrix M on a subinterval T' of T.

Suppose $d_i^2(t) \equiv 0$ on T for $0 \leq i \leq n-1$. Then the random process y(t) is at least n times differentiable (from Theorem 3 and its Corollary) and the matrices $R_i(t)$ exist for $0 \leq i \leq n$. For some t in T, redefine the integer k as

$$\text{rank } R_n(t) = k+1 .$$

Note that $k+1 \leq n-1$. Since $R_n(t)$ is continuous by assumption, rank $R_k(t)$ = rank $R_{k+1}(t) = k+1$ for all t in T', a subinterval of T. The main result of this section follows.

Theorem 6. Let $r(t,\tau)$ satisfy A1-A4, let rank $R_k(t)$ = rank $R_{k+1}(t) = k+1$ for all t in T', and assume that the component functions $\phi_1(t), \ldots \phi_n(t)$ are linearly independent on T'. Define M(t) as

$$M(t) = \Gamma_k(t)\, R_k^{-1}(t)\, \Gamma_k^t(t)$$

for t in T'. Then M(t) is admissible and therefore determines a shaping filter for $r(t,\tau)$.

Proof. Since $R_k(t)$ and $R_{k+1}(t)$ both have rank $k+1$ on T', then $y^{(k+1)}(t)$ may be expressed as a linear combination of the processes $y^{(i)}(t)$, where $0 \leq i \leq k$, for t in T'. This linear combination may be expressed as

$$\dot{Y}_k(t) = A(t)\, Y_k(t) \quad , \tag{24}$$

for some matrix $A(t)$. Post-multiplying by $Y_k^t(\tau)$ and taking the expectation of the result yields

$$\frac{\partial}{\partial t} R_k(t,\tau) = A(t)\, R_k(t,\tau) \quad , \tag{25}$$

where

$$R_k(t,\tau) = E[Y_k(t)\, Y_k^t(\tau)]^* . \tag{26}$$

Transposing (25) and noting that

$$R_k^t(t,\tau) = R_k(\tau,t)$$

yields

$$\frac{\partial}{\partial \tau} R_k(t,\tau) = R_k(t,\tau)\, A^t(\tau) \; . \tag{27}$$

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* The definition of $R_k(t)$ in (12) coincides with $R_k(t,\tau)$ defined by (26), for $\tau = t$.

For $t > \tau$,

$$R_k(t,\tau) = \Phi_k^t(t)\, \Gamma_k(\tau) ,$$

and from equation (27)

$$\frac{\partial}{\partial\tau} R_k(t,\tau) = \Phi_k^t(t)\, \dot{\Gamma}_k(\tau) = \Phi_k^t(t)\, \Gamma_k(\tau)\, A^t(\tau) . \tag{28}$$

Since the functions $\phi_1(t), \ldots, \phi_n(t)$ are linearly independent on T' by assumption, (28) yields

$$\dot{\Gamma}_k = \Gamma_k\, A^t . \tag{29}$$

Define $M(t)$ as

$$M(t) = \Gamma_k(t)\, R_k^{-1}(t)\, \Gamma_k(t) . \tag{30}$$

Clearly, $M(t)$ is defined for all t in T'. From the proof of Theorem 5, $M(t)$ is a covariance matrix and satisfies

$$\Gamma_k(t) = M(t)\, \Phi_k(t)$$

for all t in T'. In order to show that $M(t)$ is admissible, it must be established that $M(t)$ is non-decreasing. We show below that $M(t)$ is constant for t in T', and is therefore non-decreasing.

Differentiating (30) yields

$$\dot{M} = -\Gamma_k R_k^{-1} \dot{R}_k R_k^{-1} \Gamma_k^t + \dot{\Gamma}_k R_k^{-1} \Gamma_k^t + \Gamma_k R_k^{-1} \dot{\Gamma}_k^t . \tag{31}$$

However, the derivative $\dot{R}_k(t)$ may be expressed as

$$\dot{R}_k(t) = \frac{d}{dt} R_k(t,t) = \frac{\partial}{\partial t} R_k(t,\tau)\Big|_{\tau=t} + \frac{\partial}{\partial \tau} R_k(t,\tau)\Big|_{\tau=t}$$

$$= A(t)\, R_k(t) + R_k(t)\, A^t(t) . \tag{32}$$

Substituting (29) and (32) into (31) yields

$$\dot{M} = -\Gamma_k R_k^{-1} A R_k R_k^{-1} \Gamma^t - \Gamma_k R_k^{-1} R_k A^t R_k^{-1} \Gamma_k^t + \Gamma_k A^t R_k^{-1} \Gamma_k^t + \Gamma_k R_k^{-1} A \Gamma_k^t = 0 ,$$

for all t in T'. Therefore M is an admissible matrix and $r(t,\tau)$ may be expressed as

$$r(t,\tau) = \phi^t(t)\, M\, \phi(\tau)$$

Since M is constant, the shaping filter has the form

$$\dot{x} = 0$$

$$y = \phi^t x ,$$

where the initial state x_o is a random variable with covariance matrix M defined by equation (30), and $x(t) = x_o$. Furthermore, since the rank of M is k+1, the shaping filter may be reduced to one of order k+1. The assumption in the above Theorem that the functions $\phi_i(t)$ are linearly independent is not restrictive since if it is not satisfied the separation of $r(t,\tau)$ in A1 may be reduced to one of lower order by forming appropriate linear combinations of the functions $\phi_i(t)$. These linear combinations will then be linearly independent on T'.

Note finally that the class of autocorrelation functions to which Theorem 6 applies corresponds to random processes which may be represented by a truncated Karhunen-Loeve expansion.

Conclusion

This investigation was concerned with the synthesis of shaping filters corresponding to separable and differentiable autocorrelation functions. The determination of a shaping filter is based on a set of conditions (A1-A5) to be satisfied by the autocorrelation function. If the conditions are satisfied, real-valued coefficients of the shaping filter may be determined by solving a matrix Riccati differential equation of order no greater than the order of the shaping filter. If the Riccati equation cannot be defined anywhere on an interval, then an autonomous shaping filter may be determined on the interval instead.

In order to formulate the Riccati equation, it was necessary to develop and prove results concerning the mean square differentiability of a random process. The results are thus applicable to autocorrelation functions corresponding to "colored noise", i.e., possibly nonstationary random processes which do not contain white noise as a component.

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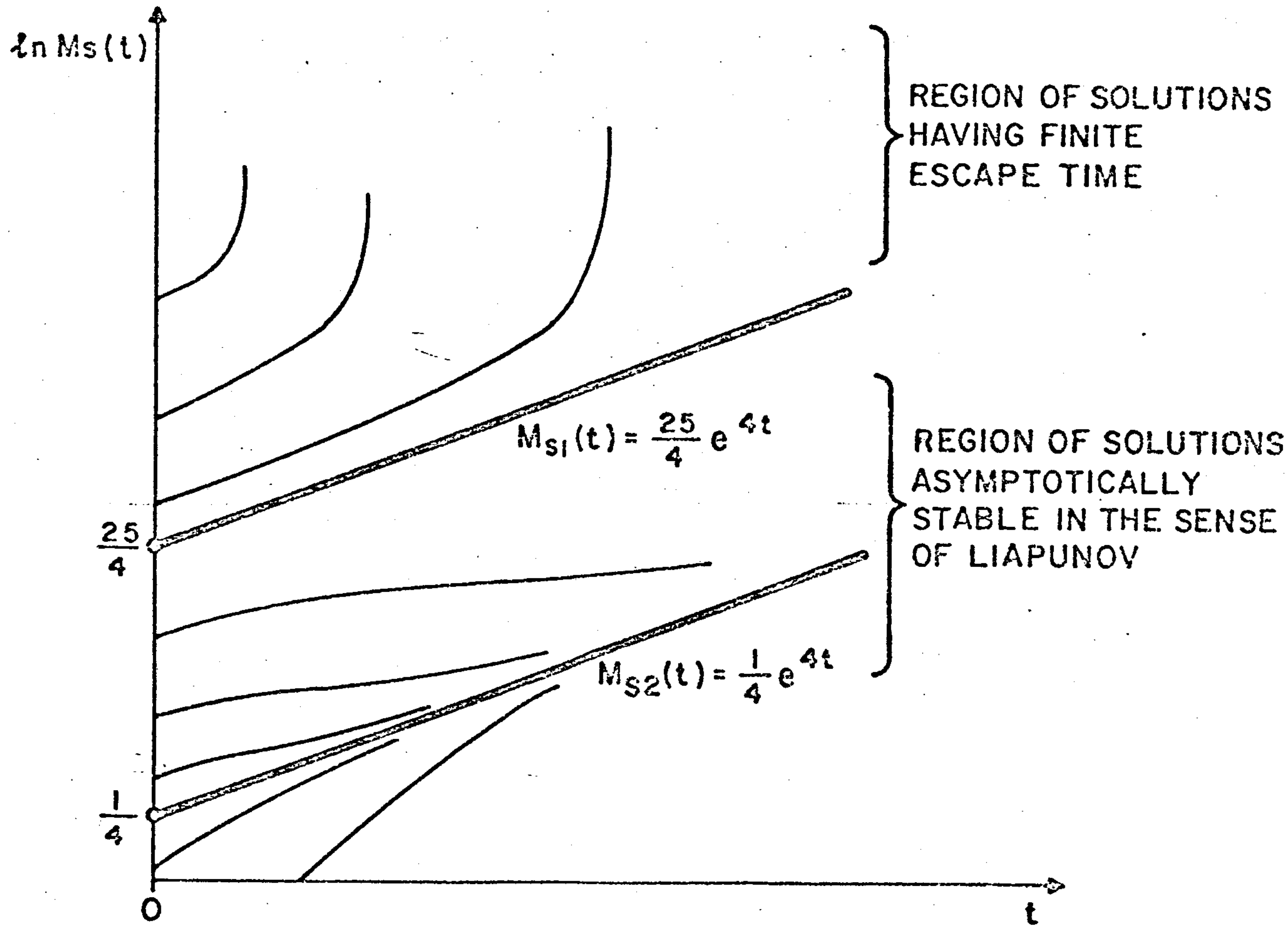

Fig. 1. Illustrating behavior of solutions of equation (22) for various initial values of $M_s(0)$. Solutions $M_{s1}(t)$ and $M_{s2}(t)$ correspond to time-invariant shaping filters.